Gödel's universe and the chronology protection conjecture

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We present a solution for the geodesic motion in Gödel's universe that provides a particular proof of Hawking's chronology protection conjecture in three-dimensional gravity theory. The solution is based upon the fact that the seven-dimensional group of the automorphisms of the Heisenberg motion group $H^1 \times U(1)$, modulo discrete sub-group \mathbf{Z} , act isometrically on the boundary of the hyperbolic three-dimensional manifold. Closed timelike curves do not exist due to the presence of a closed Cauchy-Riemann surface for chronology protection, with two mirror symmetric sets of helicoidal self-similar modules inside. The present solution is isometrically equivalent to a cylindrical gravitational monochromatic wave front.

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In 1949, Gödel [1] discovered a new solution for Einstein's field equations in which causality is violated by the existence of periodic world lines that run back into themselves. The periods are given by integrals over the proper time differential ds. These cycles are known as closed timelike curves (CTCs). According to Tipler [2] a massive infinite rotating cylinder should create a frame dragging effect in the space-time giving rise to CTCs. Tipler suggested, without proof, that in a finite rotating cylinder CTCs would arise allowing travel into the past.

The non existence of CTCs in acceptable spacetimes, was demonstrated by Deser, Jackiw and 't Hooft [3] in 1992. In the same year, Hawking [4] showed that, according to the general theory of relativity, it is impossible to build a time machine in finite regions of the spacetime without curvature singularities, due to the presence of negative energy states. Hawking posited the existence of a Cauchy-Riemann spacelike surface with a compact horizon for chronology protection, so that causality violation could never be observed from the outside. This conjecture is known as the chronology protection conjecture. For the chronology protection conjecture see also [5, 6]

Nowadays it is believed that some fundamental properties of supermassive black holes may be understood in the realm of a (2n+1)-dimensional Gödel Universe. In particular, the static Gimon-Hashimoto solution for the n=2 case [7], has attracted considerable attention associated with M-theory, pp waves, thermodynamics of black holes and strong gravitational lensing effects and has been the focus of recent research [8–10]. However, the geodesic motion in Gödel's universe is not yet completely understood even though great strides have been made over the past 55 years [11–16].

The aim of this letter is to provide, the appropriate geometrical setting to prove Hawking's chronology protection conjecture in Gödel's Universe. For our purpose it is sufficient to analyze the (2+1)-dimensional Gödel universe in a finite rotating cylinder with constant angular velocity ω , and negative cosmological constant Λ .

First of all, we show that Gödel's spacetime is invariant under local actions of the 7-dimensional group of automorphisms of the reduced Heisenberg motion group $G = H^1/\mathbf{Z} \times U(1)$. Thus, the geodesic problem is reduced to a sub-Riemannian variational problem in the cotangent space, $T^*\mathcal{M}$, of the 3-dimensional Heisenberg group, a well known problem in the realm of mathematical control theory [17, 18].

The Heisenberg Group - The Heisenberg group plays the fundamental role in diverse topics such as harmonic analysis [19], classical and quantum mechanics [20], and sub-Riemannian geometry (also known as Carnot-Charathéodory geometry) [21–24]. As it is the most simple non-Abelian nilpotent Lie group homeomorphic to the Euclidean space it offers the opportunity for generalizing the remarkable results of non-commutative harmonic analysis to soluble models of non-Abelian gauge field theories. In particular, it offers a interesting soluble scenario for Gribov's theory of quark confinement [25]

In 2000, the Heisenberg group was used to prove the possible existence of bound pairs of self-dual non-Abelian magnetic monopoles in high energy physics [26]. Recently, experiments with superconductors (SCs) doped with topological insulators (TIs), confirms the existence of such unusual configuration [27].

Recall that the (2n+1)-dimensional Heisenberg group H^n , can be realised as the multiplicative group of real $(n+2) \times (n+2)$ matrices of the form

$$A(x, y, t) = \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \equiv (z, t) \in C_{\times}^{n} \times \Re, \quad (1)$$

were we have identified $\Re^n_{\times} \times \Re^n_{\times}$ with C^n_{\times} . By setting z = x + iy and w = u + iv, H^n can be defined by the group law

$$(z,t)\cdot(w,s) = (z+w,t+s+\frac{1}{2}Im(z\cdot\overline{w})), \qquad (2)$$

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where $Im(z \cdot \overline{w}) = (u \cdot y - v \cdot x)$ is the standard symplectic form on \Re^{2n} and $z \cdot \overline{w} = z_1 \overline{w}_1 + \ldots + z_n \overline{w}_n$ is the standard Hermitian form on C^n .

The (2n+1) left invariant vector fields

$$Q_{j} = \frac{\partial}{\partial x_{j}} - \frac{1}{2}y_{j}\frac{\partial}{\partial t}; P_{j} = \frac{\partial}{\partial y_{j}} + \frac{1}{2}x_{j}\frac{\partial}{\partial t}; T = \frac{\partial}{\partial t}$$
(3)

are the generators of the Heisenberg Lie algebra h^n :

$$[P_i, P_j] = [Q_i, Q_j] = 0; [Q_i, P_j] = \delta_{ij}T.$$
 (4)

The Heisenberg motion group $G = H^n \times U(n)$, is the group of isometries of the sub-Laplacian \mathcal{L} :

$$\mathcal{L} = -\sum_{j=1}^{n} (P_j^2 + Q_j^2) = -\Delta_z - \frac{1}{4} \mathbf{r}^2 \partial_t^2 + J \partial_t, \quad (5)$$

where Δ_z is the Laplacian on C_{\times}^n , $\mathbf{r}^2 = |z|^2$ and

$$J = \sum_{j=1}^{n} (x_j \partial_{y_j} - y_j \partial_{x_j}) \tag{6}$$

is the rotation operator. In order to have a bounded space-time we must work with the compact group H^1/\mathbf{Z} . The discrete twisted sub-group \mathbf{Z} is generated by the set of three matrices $\mathbf{Z} = \{(1,0,0),(0,1,0),(0,0,1/2k\pi); k = \pm 1, \pm 2, \pm 3, ...\}$ [28].

Gödel's Universe is a rotating anisotropic homogeneous Lorentzian spacetime, in which matter takes the form of a pressure-free perfect fluid $(T_{ab} = \rho \mathbf{u}_a \mathbf{u}_b)$ where ρ is the matter density and \mathbf{u}_a are the four normalized velocity vector fields. The manifold is $\mathcal{M} = \Re^3 \times \Re$, with metric

$$ds^{2} = -c^{2}dt^{2} + dx^{2} - \frac{1}{2}e^{2\sqrt{2}kx}dy^{2} - 2e^{\sqrt{2}kx}cdtdy + du^{2}$$
(7)

where $k = \omega/c$. The relevant manifold is \Re^3 .

Einstein's field equations are satisfied if $|\mathbf{u}_0| = |\mathbf{u}_4| = 1$ and $4\pi\rho = k^2 = -\Lambda$. The constant k characterizes the angular velocity ω of the matter associated to the flow vector field \mathbf{u}_4 , Cf [29].

It is easy to verify (taking into account (ω, c)), that (7) is invariant under the automorphisms of the Heisenberg group. For $Aut(H^n)$ see [20]

Indeed for each $\alpha_i(x, y, t, u) \equiv \alpha_i(\mathbf{x}), \alpha_i \in Aut(G)$, we have :

- 1) Symplectic actions $\alpha_1(\mathbf{x}) = (y, -x, t, u)$;
- 2) Dilations $\alpha_2(\mathbf{x}) = (ax, ay, a^2t, au); k \to k/a; a > 0;$
- 3) Temporal Inversions $\alpha_3(\mathbf{x}) = (x, y, -t, u)$;
- 4) Spatial Inversions $\alpha_4(\mathbf{x}) = (-x, -y, t, -u)$;
- 5) $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ actions $\alpha_5(\mathbf{x}) = (-x, -y, t, u) = (-x, y, -t, u) = (x, -y, -t, u);$
- 6) Rotations $\alpha_6(\mathbf{x}) = (\sigma(x, y), t, u); \ \sigma \in U(1)$
- 7)Screw translations: $\alpha_7(\mathbf{x}) = (\mathbf{R}_{\theta}(\mathbf{r}), t + f(\theta), u)$, where $\mathbf{R}_{\theta} \in SO(2)$; $\mathbf{r}^2 = x^2 + y^2$ and $f(\theta)$ an harmonic function.

This invariance is the result of the fundamental principle connecting the Cauchy-Riemann geometry of the Poincaré unity ball B_{n+1} (the Siegel half-space S_{n+1}) and the sub-Riemannian geometry, which arises from the correspondence between the boundary of the Poincaré ball and the Heisenberg group [30]. So this principle enables us to identify the reduced Heisenberg motion group with the boundary of the finite (2+1) - dimensional Gödel Universe.

The most symmetric metric of the Heisenberg motion group is

$$ds^{2} = dx^{2} + dy^{2} + du[dt + g(xdy - ydx)] + du^{2}$$
 (8)

where $g = (\sqrt{2}/2)\omega$, in unities of c = 1.

The contact form $\Omega = dt + g(xdy - ydx)$ is the annihilator of the vector fields (Q, P), spans of the horizontal sub-space of the tangent space $T\mathcal{M} = H \oplus V$:

$$Q = \partial_x - gy\partial_t; P = \partial_y + gx\partial_t, \tag{9}$$

and the dual of the vector fields, (T,J), spans of the vertical sub-space :

$$T = \partial_t; \ J = g(x\partial_y - y\partial_x).$$
 (10)

That is,

$$Q(\Omega) = P(\Omega) = 0; T(\Omega) = 1; J(\Omega) = g^2(x^2 + y^2).$$
 (11)

Einstein's field equations are satisfied if |T|=|J|=1. Hence,

$$g^{2}(x^{2}+y^{2})=1 \to \omega = \frac{\sqrt{2}}{\mathbf{r}_{0}}; \to \Lambda = -\frac{2}{\mathbf{r}_{0}^{2}}.$$
 (12)

The integral lines of the vector fields (Q, P, T, J) are solutions of Einstein field equations. The solutions depend on Gödel's radius, \mathbf{r}_0 , which characterizes the mass density and the angular velocity. The circles C at t = constant are called timelike circles if $\mathbf{r} > \mathbf{r}_0$, null-circles, if $\mathbf{r} = \mathbf{r}_0$ and spacelike circles, if $\mathbf{r} < \mathbf{r}_0$, Cf [15].

In $T^*\mathcal{M}$, the vector fields (Q, P, T, J) can be represented by the functions:

$$Q = p_x - gyp_t; P = p_y + gxp_t; T = p_t; J = g(xp_y - yp_x)$$
(13)

An explicit calculation gives us the Lie algebra:

$${Q, P} = 2gT; {J, P} = gQ; {J, Q} = -gP$$
 (14)

$${P,T} = {Q,T} = {J,T} = 0.$$
 (15)

This Lie algebra arises in the Nappi-Witten model of a four-dimensional homogenous anisotropic spacetime, based on nonsemisimple Lie groups, where the metric (8) was interpreted as the metric of a gravitational monochromatic plane wave [31]. It will be shown in the following that the metric (8) gives us the wave front of a cylindrical gravitational wave, first considered by Einstein & Rosen [32] and latter described in details by Weber & Wheeler [33], and Marder [34].

The geodesics are obtained here from the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(Q^2 + P^2 + T^2) + J^2 \tag{16}$$

The Hamiltonian equations are

$$\frac{dQ}{ds} = \{\mathcal{H}, Q\} = P\{P, Q\} + 2J\{J, Q\} = -2g\lambda P, \quad (17)$$

$$\frac{dP}{ds} = \{\mathcal{H}, P\} = Q\{Q, P\} + 2J\{J, P\} = 2g\lambda Q, \quad (18)$$

$$\frac{dT}{ds} = \{\mathcal{H}, T\} = 0; \quad \frac{dJ}{ds} = \{\mathcal{H}, J\} = 0, \quad (19)$$

with the supplementary conditions

$$\frac{dx}{ds} = Q;$$
 $\frac{dy}{ds} = P;$ $\frac{dt}{ds} + g\left(y\frac{dx}{ds} - x\frac{dy}{ds}\right) = 0,$ (20)

Introducing $Q = \mathbf{r}_0 cos \psi, P = \mathbf{r}_0 sin \psi$, we have

$$-2g\lambda P = \frac{dQ}{ds} = -\mathbf{r}_0 \sin\psi \frac{d\psi}{ds}.$$
 (21)

It follows that $\psi=2g\lambda s+\phi$, where ϕ gives the initial directions of the sub-Riemannian geodesics emanate from the origin (0,0,0) of the spacetime. Integrating (20), in the interval (0,1), one obtains the sub-Riemannian wave front, that is, the manifold of the endpoints of the sub-Riemannian geodesics of length ${\bf r}_0$.

$$x = \mathbf{r}_0 \left[\frac{\cos(\theta + \phi) - \cos\phi}{\theta} \right], \tag{22}$$

$$y = \mathbf{r}_0 \left[\frac{\sin(\theta + \phi) - \sin\phi}{\theta} \right], \tag{23}$$

$$t = \mathbf{r}_0^2 \left[\frac{\theta - \sin \theta}{\theta^2} \right]. \tag{24}$$

where $\theta = 2g\lambda$, with $\lambda \in \Re$ and $0 \le \phi \le 2\pi$. These parametric equations, aided by computers, give us all relevant information about the geometry and topology of the (2+1)-dimensional Gödel's spacetime manifold:

- 1. The C circles are given by $\mathbf{r} = (\mathbf{r}_0/\theta)\sqrt{2(1-\cos\theta)}$
- 2. The horizons are localized at $\pm \pi \leq \theta \leq \pm 2\pi$, with the same radius $\mathbf{r}_h = 2\mathbf{r}_o/\pi$.
- 3. The conic singularities are localized at $\theta = \pm 2k\pi$ in the points $(\mathbf{r} = 0, t_k = \mathbf{r}_0^2/2k\pi); \quad k = \pm 1, \pm 2,$

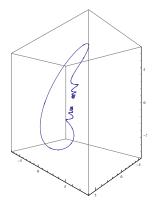


FIG. 1:



FIG. 2:

- 4. There are no timelike circles. The maximal circle is the nullcircle of radius $\mathbf{r} = \mathbf{r}_0$ in the plane t = 0 of the Cauchy-Riemann surface at $\theta = 0$
- 5. The boundary is a incomplete manifold, axially symmetric, formed by a infinite set of rectifiable curves, as shown in fig.1.
- 6. The Cauchy-Riemann surface with the array of self-similar modules is shown in fig.2.
- 7. The first module localized at $2\pi \le \theta \le 4\pi$ is shown in fig.3.

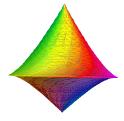


FIG. 3:

Conclusions - In this letter we have presented a finite cylindrical solution for the geodesic motion in the (2 + 1) - dimensional Gödel universe, based on the geometry and topology of the compact Heisenberg motion group. Causality is not violated, due to the existence of a well defined Cauchy-Riemann spacelike surface with compact dynamically-generated conic horizons. The spacetime is causally geodesically incomplete (according to Hawking-Penrose conditions [35]) and strongly supports Hawking's chronology protection conjecture: the laws of physics do not allow the appearance of closed timelike curves.

There is no room for negative energy (mass). As it is well know, in 3-dimensional Einstein gravity, the only possible energy measure must be topological and the Euler invariant is the only candidate. From the Gauss-Bonnet theorem we have $K\sum_{n=0}^{\infty} \int_{A_n} d\Omega = 4\pi e(\Sigma)$, where K is a constant, A_n is the area enclosed by each maximal circle, C_n , in each region of the spacetime, and $e(\Sigma)$, is the Euler number of the 2-dimensional orbifold, $\Sigma = \Sigma^- \cup \Sigma^+$. As is well known, after Thurston [36], in the case of a closed 3-manifold with geometric structure modelled on the compact Heisenberg group H^1/\mathbb{Z} , (a manifold foliated by circles), $e(\Sigma) = 1$.

The mass, at the origin of the spacetime, only affects geometry globally rather than locally. The global geometry is fixed by periodic conic singularities along the source's world line, according to the three-dimensional gravity theory of Deser, Jackiw and 't Hooft [37]. This closed homogeneous anisotropic spacetime shares many interesting properties with a physically acceptable spacetime. However, due to the notable absence of the Riemann tensor, (replaced by the non-holonomic connection, Ω), Gödel's universe based on the sub-Riemannian geometry of the Heisenberg group, is not acceptable at the large scale structure of spacetime of Einstein's general theory of relativity. Note that the rotating body (fig.2), cannot be embedded in \Re^3 , as a surface of revolution as is usual in Riemannian geometry. Indeed, the set of end points of the sub-Riemannian geodesics emanate from the origin, is a convex set in \Re^3 whose boundary has two opposed enumerable sets of conic singularities. The only really singular point is the unreachable origin, which is the closure of the spacetime.

However, this unusual configuration is acceptable at the small scale structure of quantum theory, according to the fundamental Stone-Von Neumann theorem on the unitary and irreducible representation of the Heisenberg group in the Hilbert space, which is fairly simple and well understood [19, 20].

Finally it should be noted that there is a close connection between this approach and [3], where the flat Kerr metric was used to represent the spacetime.Indeed, the three-dimensional manifold of the compact Heisenberg group, H^1/\mathbf{Z} , is a line bundle over a plane,in \Re^3 , with a prescription for identifying points in \Re^3 . This 3-dimensional topology was used to give an example of how to close a Cauchy-Riemann surface:

"The matching condition is defined only when a closed curve is followed around the source; these matchings are defined by a deficit angle and a time shift. More precisely we have a space with \Re^3/\Re topology. (a three-space with a line obstruction on the source's world line) and a prescription for identifying points." Op.Cit [3].

The matching condition is just the screw translation in \Re^3 , one of the transitive actions of the automorphism of the Heisenberg group, modulo discrete sub-group **Z**.

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